Two-loop calculation of the scaling behavior of two-dimensional forced Navier-Stokes equation

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Asymptotic properties of the solution of two-dimensional randomly forced Navier-Stokes equation with long-range correlations of the driving force are analyzed in the two-loop order of perturbation theory with the use of renormalization group. Kolmogorov constant of the energy spectrum is calculated for both the inverse energy cascade and the direct enstrophy cascade in the second order of the ε expansion.

I. INTRODUCTION

Asymptotic properties of the stochastic Navier-Stokes equation have been actively studied with the aid of renormalization group (RG) during the last two decades (for a comprehensive review, see [1]). A random force with powerlike falloff of spatial correlations has been used to maintain a stationary scale-invariant regime with a subsequent generation of an ε expansion of scaling exponents and scaling functions.

Most work has been carried out in view of application to three-dimensional turbulence; partly because of greater physical interest than in the two-dimensional case, partly due to additional divergences, which occur in two dimensions and prevent a direct use of the previously obtained general d-dimensional results. Moreover, there were some serious flaws in the early work [2] on the RG-analysis of the forced Navier-Stokes equation in two dimensions, and only recently has a consistent renormalization procedure been put forward [3,4] and used in various problems involving randomly forced incompressible fluid [5–7].

Apart from energy, in two dimensions the enstrophy (squared vorticity) is an inviscid conserved quantity quadratic in the velocity. Therefore, two self-similar regimes corresponding to an inverse energy cascade towards small wave numbers and a direct enstrophy cascade towards large wave numbers are expected to take place [8] instead of the direct energy cascade observed in three-dimensional turbulence. The energy (enstrophy) pumping leading to a steady state with the two scaling regimes may be realized in two different ways. On one hand, in numerical simulations [9] and some experiments [10] the energy and enstrophy pumping takes place on scales in between the inverse energy cascade and the enstrophy cascade. On the other hand, in atmo-spheric turbulence [11] the energy and enstrophy sources are at the outer edges of the scaling intervals, and it is not clear whether there is an energy and enstrophy sink between them [11] or they coexist [9]. In both cases the Kolmogorov spectrum of the inverse energy cascade $E(k) \propto k^{-5/3}$ for $k \ll k_I$ is observed experimentally and in the majority of simulations for wave numbers smaller than the characteristic wave number k_I of the energy pumping. For the enstrophy inertial range the existing data is not so clear-cut, although recent high-precision numerical simulations [12] and accurate analysis of the atmospheric data [13] seem to support the existence of an energy spectrum $\propto k^{-3}$ in the enstrophy cascade.

In this paper we have carried out a two-loop renormalization of the field theory generated by the two-dimensional randomly forced Navier-Stokes equation. We have taken into account the divergences specific of two dimensions in the way proposed in Ref. [3], and calculated the asymptotic expression for the energy spectrum in the second order of the ε expansion for both the energy and enstrophy inertial ranges. For convenience of calculations, we have used the stochastic Navier-Stokes equation with a subsequent dimensional regularization of divergences in two dimensions [3] instead of the specifically two-dimensional setup [4] with the stream-function description of the fluid flow.

II. RENORMALIZED FIELD THEORY FOR THE TWO-DIMENSIONAL STOCHASTIC NAVIER-STOKES EQUATION

Consider the stochastic Navier-Stokes equation for the flow of homogeneous incompressible fluid, which for the transverse components of the velocity field assumes the form

$$\partial_t v_i + P_{ij} v_l \partial_l v_j = \nu_0 \nabla^2 v_i - \xi_0 v_i + F_i \,, \qquad (2.1)$$

together with the incompressibility condition $\partial_i v_i = 0$. In Eq. (2.1) $v_i(t, \mathbf{x})$ are the coordinates of the divergenceless velocity field, ν_0 is the kinematic viscosity, ξ_0 is the coefficient of friction, and P_{ij} is the transverse projection operator ($P_{ij} = \delta_{ij} - k_i k_j / k^2$ in the wave-number space), and F_i are the coordinates of the random force. Here, and henceforth, summation over repeated indices is implied.

In experimental realizations and simulations of a two-dimensional turbulent flow energy may be consumed not

only by microscale dissipation, but also by the friction at the boundaries of the fluid layer. The friction term in Eq. (2.1) makes it possible to maintain stationary state with the anticipated inverse energy cascade towards small wave numbers and the direct enstrophy cascade towards large wave numbers, when the pumping is carried out in between the corresponding inertial ranges. The coefficient of friction is a mass term from the point of view of renormalization, therefore we put $\xi_0 = 0$ in the calculation of the renormalization constants of the solution of Eq. (2.1). As shown in Ref. [4], the friction term in Eq. (2.1) is not renormalized and thus does not affect the RG equations and the subsequent asymptotic analysis.

In the applications of the stochastic Navier-Stokes equation (2.1) to turbulence the random force is assumed to have a gaussian distribution with zero mean and the correlation function in the wave-vector space [1] of the form

$$\langle F_i(t, \mathbf{k}) F_j(t', \mathbf{k}') \rangle = P_{ij}(2\pi)^{d/2} \delta(\mathbf{k} + \mathbf{k}') \delta(t - t') d_F(k) \,. \tag{2.2}$$

The scalar kernel has a powerlike asymptotic behavior at large wave numbers:

$$d_F(k) = D_0 k^{4-d-2\varepsilon} h(m/k) \,, \tag{2.3}$$

where $h(x)$ is a well-behaved function of the dimensionless argument m/k ensuring the convergence of the inverse Fourier transform of $d_F(k)$ at small k and with the large k behavior fixed by the condition $h(0) = 1$. In a fixed dimension above two dimensions the force correlation function is not renormalized and the the kernel (2.3) remains intact. This is, however, not the case in two dimensions, in which renormalization generates additional terms $\propto k^2$ into the force correlation function. In order to deal with a multiplicatively renormalizable theory – which is convenient technically – we add this term to the correlation kernel at the outset and use, instead of the function (2.3), the modified function

$$d'_F(k) = D_{01} k^{4-d-2\varepsilon} h(m/k) + D_{02} k^2 \,. \tag{2.4}$$

The force correlation function is related to two basic physical quantities, the energy pumping rate $\mathcal{E}$ and the enstrophy pumping rate $\mathcal{B}$, as

$$\mathcal{E} = \frac{d-1}{2} \int \frac{d\mathbf{k}}{(2\pi)^d} d'_F(k) \,,$$
$$\mathcal{B} = \frac{d-1}{2} \int \frac{d\mathbf{k}}{(2\pi)^d} k^2 d'_F(k) \tag{2.5}$$

in d-dimensional space, which allows to connect the "coupling constant" D_{01} with the pumping rate in the corresponding asymptotic region.

We cast the stochastic problem (2.1), (2.2), (2.4) into a field theory with the De-Dominicis-Janssen "action" in the usual manner [1]. An analysis of UV divergences with the use of Galilei invariance, causality and symmetries of the model allows to write the renormalized action in the form

$$S = \frac{1}{2} \int \frac{dt d\mathbf{k}}{(2\pi)^d} \tilde{\mathbf{v}}$$
$$\times \left[g_1 \nu^3 \mu^{2\varepsilon} (k^2)^{1-\delta-\varepsilon} h(m/k) + g_2 \nu^3 \mu^{-2\delta} Z_2 k^2 \right] \tilde{\mathbf{v}} \quad (2.6)$$
$$+ \int dt d\mathbf{x} \, \tilde{\mathbf{v}} \cdot \left[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu Z_1 \nabla^2 \mathbf{v} \right] \,,$$

where μ is the scale-setting parameter of the renormalized model and $2\delta = d - 2$ is the parameter of dimensional regularization. Only two renormalization constants Z_1 and Z_2 are needed to absorb the UV divergences of the model in two dimensions. To avoid excessive notation, we have used the same symbols for both the fields and their Fourier transforms in (2.6).

Renormalized parameters of the action (2.6) are defined by

$$\nu_0 = \nu Z_1 \,, \tag{2.7}$$
$$D_{01} = g_1 \nu^3 \mu^{2\varepsilon} Z_1^{-3} \,, \tag{2.8}$$
$$D_{02} = g_2 \nu^3 \mu^{-2\delta} Z_2 Z_1^{-3} \,. \tag{2.9}$$

We have used a combination of dimensional and analytic regularization with the parameters ε and $2\delta = d - 2$. As a consequence, the UV divergences appear as poles in linear combinations of the regularizing parameters. Normalization has been fixed by the choice of the minimal subtraction (MS) scheme [14].

III. RENORMALIZATION-GROUP EQUATIONS AND FIXED POINTS

We set up the notation and basic equations for the spatial Fourier transform of the pair correlation function of the random velocity field

$$W_{mn}(t_1 - t_2, \mathbf{k}; D_{01}, D_{02}, \nu_0) = \int \frac{d^d \mathbf{x}_1}{(2\pi)^d}$$
$$\times \langle v_m(t_1, \mathbf{x}_1) v_n(t_2, \mathbf{x}_2) \rangle e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} \,, \tag{3.1}$$

because this quantity is directly connected to the energy spectrum through the relation $\langle v_n(t, \mathbf{x}) v_n(t, \mathbf{x}) \rangle = 2 \int_0^\infty E(k) dk$.

Independence of the unrenormalized pair correlation function

$$W_{mn}(t, \mathbf{k}; D_{01}, D_{02}, \nu_0) = W_{mn}^R(t, \mathbf{k}; g_1, g_2, \nu, \mu)$$

of the velocity field $\mathbf{v}$ of the scale-setting parameter μ gives rise to the basic RG equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_1 \frac{\partial}{\partial g_1} + \beta_2 \frac{\partial}{\partial g_2} - \gamma_1 \nu \frac{\partial}{\partial \nu} \right] W_{mn}^R = 0 \tag{3.2}$$

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for the renormalized correlation function W_{mn}^R. The coefficient functions of Eq. (3.2) β_1, β_2, and γ_1 are expressed in terms of logarithmic derivatives of the renormalization constants. We use the definitions

$$\gamma_i = \mu \frac{\partial \ln Z_i}{\partial \mu}\bigg|_0 \, , \quad \beta_i = \mu \frac{\partial g_i}{\partial \mu}\bigg|_0 \, , \tag{3.3}$$

where $i = 1, 2$, and the subscript "0" refers to partial derivatives taken at fixed values of the bare parameters.

It is convenient to express the correlation function through a dimensionless scalar function R. In two-dimensional space we define this function through the relation

$$W_{mn}^R(t, \mathbf{k}; g_1, g_2, \nu) = \frac{1}{2} g_1 \nu^2 P_{mn}(\mathbf{k}) R(\tau, s; g_1, g_2) \, , \tag{3.4}$$

where $s = k/\mu$ is the dimensionless wave number, and $\tau = t\nu k^2$ the dimensionless time. Solving Eqs. (3.2), (3.3) by the method of characteristics we obtain the correlation function in the form

$$W_{mn}^R(t, \mathbf{k}; g_1, g_2, \nu) = \frac{1}{2} \overline{g}_1 \overline{\nu}^2 P_{mn}(\mathbf{k}) \, R\left(tk^2 \overline{\nu}, 1; \overline{g}_1, \overline{g}_2\right) \, , \tag{3.5}$$

where $\overline{g}_i$ are the solution of the Gell-Mann-Low equations:

$$\frac{d\overline{g}_i}{d\ln s} = \beta_i\left[\overline{g}_1, \overline{g}_2\right] \, , \qquad i = 1, 2 \, , \tag{3.6}$$

and $\overline{\nu}$ is the running coefficient of viscosity

$$\overline{\nu} = \nu e^{-\int_1^s dx\, \gamma_1(\overline{g}_1(x), \overline{g}_2(x))/x} \, . \tag{3.7}$$

Writing the latter in terms of the unrenormalized (physical) parameters and the running coupling constant $\overline{g}_1$ [1] as

$$\overline{\nu} = \left(\frac{D_{01}}{\overline{g}_1}\right)^{1/3} k^{-2\varepsilon/3} \, , \tag{3.8}$$

we arrive at the expression

$$W_{mn}^R(t, \mathbf{k}; g_1, g_2, \nu) = \frac{1}{2}\, \overline{g}_1^{1/3} D_{01}^{2/3} k^{-4\varepsilon/3}$$

$$\times P_{mn}(\mathbf{k}) R\left((D_{01}/\overline{g}_1)^{1/3} k^{2-2\varepsilon/3} t, 1; \overline{g}_1, \overline{g}_2\right) \, . \tag{3.9}$$

For the β-functions:

$$\begin{aligned} \beta_1 &= g_1(-2\varepsilon + 3\gamma_1) \, , \\ \beta_2 &= g_2(-\gamma_2 + 3\gamma_1) \, , \end{aligned} \tag{3.10}$$

a tedious two-loop calculation, with the use of the step function $h(m/k) = \theta(k - m)$ in the kernel (2.4), and dimensional regularization, yields

$$\gamma_1 = u_1 + u_2 + u_1^2\left(\frac{1}{2}\alpha + \frac{3}{4} - r\right)$$

$$+ u_1 u_2\left(\alpha + \frac{3}{2} - 2r\right) + u_2^2\left(-\frac{1}{2} - r\right)$$

$$\gamma_2 = \frac{(u_1 + u_2)^2}{u_2} + \frac{u_1^3(1 - r)}{u_2} \tag{3.11}$$

$$+ u_1^2\left(\frac{5}{2}\alpha + \frac{31}{4} - 3r\right)$$

$$+ u_1 u_2\left(\frac{5}{2}\alpha + \frac{25}{4} - 3r\right) + u_2^2\left(-\frac{1}{2} - r\right) \, ,$$

where

$$u_1 = \frac{g_1}{32\pi} \, , \qquad u_2 = \frac{g_2}{32\pi} \, . \tag{3.12}$$

The constant $r = -0.1685$ in Eq. (3.11) comes from a numerical calculation of those parts of two-loop graphs, for which analytic results were not feasible. The constant $\alpha = C - \ln(4\pi) = -1.9538$ (C is Euler's constant) is brought about by a $2\delta = d - 2$ expansion of the geometric factor $S_d/(2\pi)^d = 2/[\Gamma(d/2)(4\pi)^{d/2}] = \frac{1}{2\pi}[1 + \alpha\delta + O(\delta^2)]$. The method of calculation is essentially the same as was used for the d-dimensional case in Ref. [15]

At one-loop order the γ-functions (3.11) are exactly the same as those of the d-dimensional Navier-Stokes equation in two dimensions [3]. They also coincide with the expressions obtained directly in two dimensions for the corresponding stochastic vorticity equation [4]. Thus, we think that for calculation of the coefficient functions of the renormalization group equation the results of the d-dimensional model in the two-parameter expansion may be applied directly to the two-dimensional case. There might be some discrepancies in calculations involving composite operators due to different symmetries in two-dimensional and general d-dimensional cases, but we do no calculate anything like that here.

The fixed points are determined by the system of equations $\beta_1 = \beta_2 = 0$. From the solution of Eqs. (3.6) near a fixed point it follows that the fixed point is infrared stable, when the matrix $\omega_{nm} = \partial_n \beta_m$ is positive definite. If $\varepsilon < 0$, then the trivial fixed point: $u_1^* = u_2^* = 0$ is infrared stable. The anomalous asymptotic behavior of the model at small wave numbers is governed by the nontrivial fixed point

$$u_1^* = \frac{4}{9}\varepsilon - \frac{2}{27}(2\alpha + 5 - 4r) \, ,$$

$$u_2^* = \frac{2}{9}\varepsilon - \frac{4}{81}(\alpha - 2 - 3r) \, , \tag{3.13}$$

at which the eigenvalues of the stability matrix are

$$\omega_{1,2} = \left(\frac{4}{3} \pm i\frac{2\sqrt{2}}{3}\right)\varepsilon$$

$$+ \left(-\frac{2}{3} - \frac{4}{9}r \pm i\frac{3 - 2r}{9}\sqrt{2}\right)\varepsilon^2 \, . \tag{3.14}$$

Note that these eigenvalues are independent of α. The real parts of both eigenvalues (3.14) are positive and the fixed point (3.13) infrared stable, when $\varepsilon > 0$.

IV. TWO-LOOP CALCULATION OF KOLMOGOROV CONSTANTS

The connection between the energy spectrum $E(k)$ and the equal-time correlation function of the velocity field, $\langle v_n(\mathbf{x})v_n(\mathbf{x})\rangle = 2\int_0^\infty E(k)dk$, in the two-dimensional wave-vector space amounts to

$$E(k) = \frac{k}{4\pi}W_{nn}(0, \mathbf{k}). \tag{4.1}$$

From (3.9) the asymptotic expression

$$E(k) = g_*^{1/3}D_{01}{}^{2/3}\frac{k^{1-4\varepsilon/3}}{8\pi}R\left(0, 1; g_1^*, g_2^*\right) \tag{4.2}$$

follows, when $k \to 0$. Here, $g_1^* = 32\pi u_1^*$, $g_2^* = 32\pi u_2^*$ are the values of the coupling constants at the infrared-stable fixed point (3.13).

The relations (2.5) allow to express the parameters D_{01} and D_{02} in terms of the energy (or enstrophy) pumping rates $\mathcal{E}$ ($\mathcal{B}$). Integrating over the wave-number shell $m < k < \Lambda$ with $h(x) = 1$ in the kernel (2.4) we obtain, in the limit of widely separated upper and lower wave-number limits,

$$\mathcal{E} = \frac{D_{01}}{8\pi}\frac{\Lambda^{2(2-\varepsilon)}}{2-\varepsilon} + \frac{D_{02}}{16\pi}\Lambda^4, \tag{4.3}$$

$$\mathcal{B} = \frac{D_{01}}{8\pi}\frac{\Lambda^{2(3-\varepsilon)}}{3-\varepsilon} + \frac{D_{02}}{24\pi}\Lambda^6. \tag{4.4}$$

The spectrum (4.2) should be independent of the details of the energy pumping, i.e. independent of the upper cutoff Λ in the range $m \ll k \ll \Lambda$. According to the relation (4.3), this goal is achieved by the choice $D_{02} = 0$ and $\varepsilon = 2$ for the anticipated inverse energy cascade. The relation (4.4), in turn, shows that the choice $D_{02} = 0$ and $\varepsilon = 3$ leads to scale-invariant behavior for the direct enstrophy cascade. In both cases it should be borne in mind that the bare coupling constant D_{02} is technically a book-keeping parameter reflecting the necessity of the introduction of the short-range term in the correlation function of the random force. Physically, it could be related to the intensity of thermal fluctuations, which, however, are irrelevant in the energy balance of stationary developed turbulence.

The Kolmogorov constants are determined from the asymptotic relations

$$E(k) = C(\varepsilon)\mathcal{E}^{2/3}k^{-5/3}\left(\frac{\Lambda}{k}\right)^{4(\varepsilon-2)/3}, \tag{4.5}$$

$$E(k) = C'(\varepsilon)\mathcal{B}^{2/3}k^{-3}\left(\frac{\Lambda}{k}\right)^{4(\varepsilon-3)/3}. \tag{4.6}$$

Thus, the choice $\varepsilon = 2$ in (4.3) renders the spectrum (4.2) completely scale-invariant with the Kolmogorov exponents corresponding to the energy cascade, whereas the substitution $\varepsilon = 3$ in (4.4) leads to scale-invariant behavior in the enstrophy cascade.

The Kolmogorov constants may be calculated in the ε expansion from (4.2), (4.5) and (4.6). The present two-loop calculation allows to find correction terms to the previosly found [4] expressions. The result is

$$C(\varepsilon) = 2\cdot3^{1/3}\varepsilon^{1/3}\left[1 + \frac{2}{9}(1+r)\varepsilon\right], \tag{4.7}$$

$$C'(\varepsilon) = 3\cdot2^{1/3}\varepsilon^{1/3}\left[1 + \frac{3+2r}{9}\varepsilon\right]. \tag{4.8}$$

For $\varepsilon = 2$ we obtain from (4.7) $C = 4.977$. The closure model leads to the prediction $C = 6.69$ [8]. Results of numerical simulations vary from $C = 2.9$ [16] to $C \sim 14$ [17]. Experimental results [10] yield the range $3 < C < 7$. The result obtained here is thus in better agreement with other available data than the leading-order value of the ε expansion $C = 3.634$ obtained in Ref. [4].

For $\varepsilon = 3$ the value $C' = 10.29$ obtained from (4.8) is significantly larger than the leading-order value $C' = 5.451$ of Ref. [4] and the discrepancy between the present result and the closure-model prediction $C' = 2.626$ [8] is larger. However, from a more detailed analysis of the model it may be concluded that calculation of the constant C' is not unambiguous. The value obtained from (4.8) corresponds to the case, in which the coefficient of friction $\xi_0 = 0$. If, however, this coefficient is retained, then the asymptotic expression for the energy spectrum is [4]

$$E(k) = g_*^{1/3}D_{01}{}^{2/3}\frac{k^{1-4\varepsilon/3}}{8\pi}$$
$$\times R\left(0, 1; g_1^*, g_2^*, \frac{(g_1^*)^{1/3}\xi_0}{D_{01}^{1/3}}k^{-(2-2\varepsilon/3)}\right) \tag{4.9}$$

instead of (4.2). The energy spectrum (4.9) is scale-invariant for $\varepsilon = 3$ regardless of the value ξ_0. Therefore, the proportionality constant in the scaling law seems to be nonuniversal in the enstrophy inertial range, and depends on the properties of large-scale dissipation [4]. Recent spectral closure analysis and numerical simulations have led to similar conclusions [18].

V. CONCLUSION

In this paper we have carried out a two-loop renormalization of the randomly forced Navier-Stokes equation with long-range correlated random force in two dimensions in view of two different patterns of scale-invariant asymptotic behavior.

We have calculated the Kolmogorov constant for a powerlike asymptotic energy spectrum $\propto k^{-5/3}$ of the random velocity field in the inertial range of the inverse energy cascade in the second order of an ε expansion with the result $C = 4.977$, which is in reasonable agreement with other available experimental and theoretical data.

We have also calculated the Kolmogorov constant for the spectrum $\propto k^{-3}$ in the inertial range of the direct enstrophy cascade with the second-order result $C' = 10.29$ with a larger deviation than at the leading order from results obtained by other methods. However, explicit asymptotic expressions for the pair correlation function of the random velocity field obtained in the present approach strongly indicate that the Kolmogorov constant in the enstrophy cascade is not universal, but depends on the enstrophy dissipation due to large-scale friction.

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